

TWISTED ALEXANDER INVARIANTS AND HYPERBOLIC VOLUME

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ABSTRACT. We give a volume formula of hyperbolic knot complements using twisted Alexander invariants.

1. INTRODUCTION

The purpose of this note is to give a formula of the hyperbolic volume of a knot complement using twisted Alexander invariants.

A twisted Alexander polynomial was first defined in [3] for knots in the 3-sphere, and Wada ([14]) generalized this work and showed how to define a twisted Alexander polynomial given only a presentation of a group and representations to \mathbb{Z} and $\mathrm{GL}(V)$ where V is a finite dimensional vector space over a field. In [2], Kitano proved that in the case of knot groups the twisted Alexander polynomial can be regarded as a Reidemeister torsion.

Let M be a compact and oriented 3-manifold whose interior admits a finite volume hyperbolic structure. Porti ([8]) has investigated the Reidemeister torsion of M associated with the adjoint representation $\mathrm{Ad} \circ \mathrm{Hol}_M$ of its holonomy representation $\mathrm{Hol}_M : \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$, and then Yamaguchi showed in [11] a relationship between the Porti's Reidemeister torsion and the twisted Alexander invariant explicitly.

Müller's work ([7]) provides the relation between the Ray-Singer torsion and the hyperbolic volume of a compact hyperbolic 3-manifold. By another work ([6]) of Müller on the equivalence between the Reidemeister torsion and the Ray-Singer torsion for unimodular representations, we know the hyperbolic volume of a compact 3-manifold can be expressed using a Reidemeister torsion. After the works, Menal-Ferrer and Porti ([5]) obtained a formula of the volume of a cusped hyperbolic 3-manifold M using 'Higher-dimensional Reidemeister torsion invariants', which are associated with representations $\rho_n : \pi_1(M) \rightarrow \mathrm{SL}(n, \mathbb{C})$ corresponding to the holonomy representation $\mathrm{Hol}_M : \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ (see Section 3 for the detail).

In this note, we show that the Yamaguchi's method in [10, 11] is applicable to Higher-dimensional Reidemeister torsion invariants, so that we have a formula of the hyperbolic volume of a knot complement using twisted Alexander invariants. Let $\Delta_{K, \rho_n}(t)$ be a twisted Alexander invariant of Wada's notation ([14]). For the integer $k(> 1)$, set $\mathcal{A}_{K, 2k}(t) := \frac{\Delta_{K, \rho_{2k}}(t)}{\Delta_{K, \rho_2}(t)}$ and $\mathcal{A}_{K, 2k+1}(t) := \frac{\Delta_{K, \rho_{2k+1}}(t)}{\Delta_{K, \rho_3}(t)}$.

Theorem 1.1. *Let K be a hyperbolic knot in the 3-sphere. Then*

$$\lim_{k \rightarrow \infty} \frac{\log |\mathcal{A}_{K, 2k+1}(1)|}{(2k+1)^2} = \lim_{k \rightarrow \infty} \frac{\log |\mathcal{A}_{K, 2k}(1)|}{(2k)^2} = \frac{\mathrm{Vol}(K)}{4\pi}.$$

In the last section, we give some calculations for the figure eight knot. The details, including link case, will be given elsewhere.

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2. REIDEMEISTER TORSIONS AND TWISTED ALEXANDER INVARIANTS

Following [9] and [11], we review some definitions and conventions in this section.

Let \mathbb{F} be a field and $C_* = (C_*, \partial)$ a chain complex of finite dimensional \mathbb{F} -vector spaces:

$$0 \rightarrow C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \rightarrow 0.$$

For each i , we denote by $B_i = \text{Im}(C_{i+1} \xrightarrow{\partial} C_i)$, $Z_i = \ker(C_i \xrightarrow{\partial} C_{i-1})$, and the homology is denoted by $H_i = Z_i/B_i$. By the definition of Z_i , B_i and H_i , we obtain the following exact sequence:

$$\begin{aligned} 0 \rightarrow Z_i \rightarrow C_i \xrightarrow{\partial} B_{i-1} \rightarrow 0, \\ 0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0. \end{aligned}$$

Let \tilde{B}_{i-1} be a lift of B_{i-1} to C_i , and \tilde{H}_i a lift of H_i to Z_i . Then we can decompose C_i as follows:

$$\begin{aligned} C_i &= Z_i \oplus \tilde{B}_{i-1} \\ &= B_i \oplus \tilde{H}_i \oplus \tilde{B}_{i-1}. \end{aligned}$$

Let c^i be a basis for C_i and \mathbf{c} the collection $\{c^i\}_{i \geq 0}$. Similarly, let h^i be a basis for H_i , if nonzero, and \mathbf{h} the collection $\{h^i\}_{i \geq 0}$. We choose b^i a basis of B_i . Let \tilde{b}^{i-1} be a lift of b^{i-1} to C_i , and \tilde{h}^i a lift of h^i to Z_i , then we have a new basis $b^i \sqcup \tilde{b}^{i-1} \sqcup \tilde{h}^i$ of C_i , where \sqcup means a disjoint union. We denote by $[b^i, \tilde{b}^{i-1}, \tilde{h}^i/c^i]$ the determinant of the transformation matrix from the basis c^i to $b^i \sqcup \tilde{b}^{i-1} \sqcup \tilde{h}^i$.

Definition 2.1. The torsion of the chain complex C_* with basis \mathbf{c} and \mathbf{h} for H_i is:

$$\text{tor}(C_*, \mathbf{c}, \mathbf{h}) = \prod_{i=0}^d [b^i, \tilde{b}^{i-1}, \tilde{h}^i/c^i]^{(-1)^{i+1}} \in \mathbb{F}^*/\{\pm 1\}$$

It is known that $\text{tor}(C_*, \mathbf{c}, \mathbf{h})$ is independent of the choice of b^i and the lifts \tilde{b}^{i-1} and \tilde{h}^i .

Remark 2.2. In [5], they use $(-1)^i$ instead of $(-1)^{i+1}$ in Definition 2.1. Then the sign of the right hand side of the equation in Theorem 7.1 in [5] becomes opposite. See Remark 2.2 and Theorem 4.5 in [9].

Let W be a finite CW-complex, and $\rho : \pi_1(W, *) \rightarrow \text{SL}(n, \mathbb{F})$ a representation of its fundamental group. Consider the chain complex of vector spaces

$$C_*(W, \rho) := \mathbb{F}^n \otimes_{\rho} C_*(\tilde{W}; \mathbb{Z})$$

where $C_*(\tilde{W}, \mathbb{Z})$ denotes the simplicial complex of the universal covering of W and \otimes_{ρ} means that one takes the quotient of $\mathbb{F}^n \otimes_{\mathbb{Z}} C_*(\tilde{W}; \mathbb{Z})$ by \mathbb{Z} -module generated by

$$\rho(\gamma)^{-1}v \otimes c - v \otimes \gamma \cdot c.$$

Here, $v \in \mathbb{F}^n$, $\gamma \in \pi_1(W, *)$ and $c \in C_*(\tilde{W}; \mathbb{Z})$. Namely,

$$v \otimes \gamma \cdot c = \rho(\gamma)^{-1}v \otimes c \quad \forall \gamma \in \pi_1(W, p).$$

The boundary operator is defined by linearity and $\partial(v \otimes c) = (\text{Id} \otimes \partial)(v \otimes c) = v \otimes \partial c$. We denote by $H_*(W, \rho)$ the homology of this complex.

Let $\{v_1, \dots, v_n\}$ be a basis of \mathbb{F}^n and let $c_1^i, \dots, c_{k_i}^i$ denote the set of i -dimensional cells of W . We take a lift \tilde{c}_j^i of the cell c_j^i in \widetilde{W} . Then, for each i , $\tilde{c}^i = \{\tilde{c}_1^i, \dots, \tilde{c}_{k_i}^i\}$ is a basis of the $\mathbb{Z}[\pi_1(W)]$ -module $C_i(\widetilde{W}; \mathbb{Z})$. Thus we have the following basis of $C_i(W, \rho)$:

$$c^i = \{v_1 \otimes \tilde{c}_1^i, v_2 \otimes \tilde{c}_1^i, \dots, v_n \otimes \tilde{c}_{k_i}^i\}.$$

Suppose $H_i(W, \rho) \neq 0$, and h^i be a basis of $H_i(W; \rho)$. We denote by \mathbf{h} the basis $\{h^0, \dots, h^{\dim W}\}$ of $H_*(W, \rho)$. Then $\text{tor}(C_*(W, \rho), \mathbf{c}, \mathbf{h}) (\in \mathbb{F}^*/\{\pm 1\})$ is well defined. Note that it does not depend on the lifts of the cells \tilde{c}^i since $\det \rho = 1$. Further, if the Euler characteristic of W is equal to zero (e.g. the case that W corresponds to a knot exterior), it does not depend on the choice of a basis $\{v_1, \dots, v_n\}$ (cf. Lemma 2.4.2 [11]).

Remark 2.3. The Reidemeister torsion is independent of the choice of a base point b of the fundamental group $\pi_1(W, *)$. Furthermore, it is known that the Reidemeister torsion is an invariant under subdivision of the cell decomposition of W with ρ -coefficients up to factor ± 1 .

Remark 2.4. Let K be a knot in the 3-sphere S^3 and $M_K = S^3 - \text{Int}N(K)$. We denote by $G(K)$ the fundamental group of M_K . From the result of Waldhausen [15], the Whitehead group $\text{Wh}(G(K))$ is trivial. In such case, the Reidemeister torsion does not depend on the choice of its CW-structure. Suppose $H_*(M_K, \rho) = 0$. Then the Reidemeister torsion does not depend on $\mathbf{h} = \emptyset$. In this case we denote by $\text{tor}(M_K, \rho)$ the Reidemeister torsion.

Let α be a surjective homomorphism from $\pi_1(W, *)$ to the multiplicative group $\langle t \rangle$. Instead of a representation $\rho : \pi_1(W, *) \rightarrow \text{SL}(n, \mathbb{F})$, consider the twisted representation:

$$\alpha \otimes \rho : \pi_1(W, *) \rightarrow \text{GL}(\mathbb{F}(t)),$$

where $\mathbb{F}(t)$ is the field of fraction of the polynomial ring $\mathbb{F}[t]$. By the same method as above, we can define $\text{tor}(C_*(W, \alpha \otimes \rho), \mathbf{1} \otimes \mathbf{c}, \mathbf{h}) (\in \mathbb{F}^*(t)/\{\pm t^{n\mathbb{Z}}\})$. As the determinant is not one, there is an independency factor t^{nm} , for some integer m . More preciously, we define:

$$C_*(W, \alpha \otimes \rho) = \mathbb{F}(t) \otimes_{\mathbb{F}} \mathbb{F}^n \otimes_{\rho} C_*(\widetilde{W}; \mathbb{Z}),$$

where the action is given by $f \otimes v \otimes (\gamma \cdot c) = f \cdot t^{\alpha(\gamma)} \otimes \rho(\gamma)^{-1} v \otimes c$ for $\gamma \in \pi_1(W, p)$. The boundary operator is defined by linearity and $\partial(f \otimes v \otimes c) = f \otimes v \otimes \partial c$.

Kitano ([2]) investigated the relationship between the Reidemeister torsions and the twisted Alexander invariants for knots. Namely, he proved that:

Theorem 2.5 ([2]). *Let K be a knot in the 3-sphere S^3 and $M_K = S^3 - \text{Int}N(K)$. Suppose ρ is a non-trivial representation such that $H_*(M_K, \rho) = 0$. Then, $H_*(M_K, \alpha \otimes \rho) = 0$ and $\text{tor}(M_K, \alpha \otimes \rho) = \Delta_{K, \rho}(t)$, where $\Delta_{K, \rho}(t)$ is the twisted Alexander invariant.*

See also Theorem 2.13 in [9]. The twisted Alexander invariant can be computed using the Fox calculus [1, 2, 14].

3. REPRESENTATIONS OF THE FUNDAMENTAL GROUPS OF HYPERBOLIC 3-MANIFOLDS

Let M be an oriented, complete, hyperbolic 3-manifold of finite volume. Then M has the holonomy representation: $\text{Hol}_M : \pi_1(M, *) \rightarrow \text{Isom}^+ \mathbb{H}^3$, where $\text{Isom}^+ \mathbb{H}^3$ is the orientation preserving isometry group of hyperbolic 3-space \mathbb{H}^3 . Using the upper half-space

model, $\text{Isom}^+ \mathbb{H}^3$ is identified with $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C}) / \{\pm 1\}$. It is known that Hol_M can be lifted to $\text{SL}(2, \mathbb{C})$, and such lifts are in canonical one-to-one correspondence with spin structures on M . Thus, attached to a fixed spin structure η on M , we get a representation:

$$\text{Hol}_{(M, \eta)} : \pi_1((M, \eta), *) \rightarrow \text{SL}(2, \mathbb{C}).$$

Let W be a finite CW-complex and ρ a representation of $\pi_1(W, *)$ to $\text{SL}(2, \mathbb{C})$. Then the pair (\mathbb{C}^2, ρ) is an $\text{SL}(2, \mathbb{C})$ -representation of $\pi_1(W, *)$ by the standrd action $\text{SL}(2, \mathbb{C})$ to \mathbb{C}^2 . It is known that the pair of the symmetric product $\text{Sym}^{n-1}(\mathbb{C}^2)$ and the induced action by $\text{SL}(2, \mathbb{C})$ gives an n -dimensional irreducible representation of $\text{SL}(2, \mathbb{C})$. More precisely, let V_n be the vector space of homogeneous polynomials on \mathbb{C}^2 with degree $n - 1$, that is,

$$V_n = \text{span}_{\mathbb{C}} \langle x^{n-1}, x^{n-2}y, \dots, xy^{n-2}, y^{n-1} \rangle.$$

Then the symmetric product $\text{Sym}^{n-1}(\mathbb{C}^2)$ can be identified with V_n and the action of $A \in \text{SL}(2, \mathbb{C})$ is expressed as

$$A \cdot p \begin{pmatrix} x \\ y \end{pmatrix} = p(A^{-1} \begin{pmatrix} x \\ y \end{pmatrix})$$

where $p \begin{pmatrix} x \\ y \end{pmatrix}$ is a homogeneous polynomial and the right hand side is determined by the action of A^{-1} on the column vector as a matrix multiplication. We denote by (V_n, σ_n) the representation given by this action of $\text{SL}(2, \mathbb{C})$ where σ_n means the homomorphism from $\text{SL}(2, \mathbb{C})$ to $\text{GL}(V_n)$. It is known that each representation (V_n, σ_n) turns into an irreducible $\text{SL}(n, \mathbb{C})$ -representation of $\text{SL}(2, \mathbb{C})$ and that every irreducible n -dimensional representation of $\text{SL}(2, \mathbb{C})$ is equivalent to (V_n, σ_n) . Composing $\text{Hol}_{(M, \eta)}$ with σ_n , we obtain the following representation:

$$\rho_n : \pi_1((M, \eta), *) \rightarrow \text{SL}(n, \mathbb{C}).$$

In the following section, we will discuss Reidemeister torsions associated with this representation ρ_n . Note that there are several computations of the Reidemeister torsions associated with σ_{2k} in [12, 13].

4. THE RESULTS OF MENAL-FERRER AND PORTI

In this note, we focus on a knot complement. We introduce the results of Menal-Ferrer and Porti [4, 5] in this setting.

Let K be a hyperbolic knot in the 3-sphere S^3 , that is, $S^3 - K$ is an oriented, complete, finite-volume hyperbolic manifold with only one cusp. Then, $S^3 - K$ may be regarded as the interior of a compact manifold M_K such that $\partial M_K = T$ where T is homeomorphic to a torus T^2 . In what follows, we consider the compact manifold M_K instead of $S^3 - K$.

By Corollary 3.7 in [4], we have that $\dim_{\mathbb{C}} H^i(M_K, \rho_n) = 0$ ($i = 0, 1, 2$) if n is even, and that $\dim_{\mathbb{C}} H^0(M_K, \rho_n) = 0$, $\dim_{\mathbb{C}} H^1(M_K, \rho_n) = \dim_{\mathbb{C}} H^2(M_K, \rho_n) = 1$ if n is odd. Further, in [5], Menal-Ferrer and Porti proved the following. (Note that Poincaré duality with coefficients in ρ_n holds (Corollary 3.7 in [5]).)

Proposition 4.1 (Proposition 4.6 in [5]). *Suppose that $H_*(T; \rho_n) \neq 0$. Let $G < \pi_1(M_K, *)$ be some fixed realization of the fundamental group of T as a subgroup of $\pi_1(M_K, *)$. Choose a non-trivial cycle $\theta \in H_1(T; \mathbb{Z})$, and a non-trivial vector $v \in V_n$ fixed by $\rho_n(G)$. Then the following holds:*

- (1) *A basis for $H_1(M_K, \rho_n)$ is given by $i_*([v \otimes \tilde{\theta}])$.*
- (2) *A basis for $H_2(M_K, \rho_n)$ is given by $i_*([v \otimes \tilde{T}])$.*

Here, $i : T \hookrightarrow M_K$ denotes the inclusion.

Set $h^1 = i_*([v \otimes \tilde{\theta}])$, $h^2 = i_*([v \otimes \tilde{T}])$, and $\mathbf{h} = \{h^1, h^2\}$. On the other hand, Menal-Ferrer and Porti (Theorem 0.2 in [4]) proved that $H^*(M_K, \rho_{2k}) = 0$ for $k \geq 1$. Therefore, we may define the following quotients.

$$\begin{aligned}\mathcal{T}_{2k+1}(M_K, \eta) &:= \frac{\text{tor}(M_K, \rho_{2k+1}, \mathbf{h})}{\text{tor}(M_K, \rho_3, \mathbf{h})} \in \mathbb{C}^*/\{\pm 1\} \\ \mathcal{T}_{2k}(M_K, \eta) &:= \frac{\text{tor}(M_K, \rho_{2k})}{\text{tor}(M_K, \rho_2)} \in \mathbb{C}^*/\{\pm 1\}\end{aligned}$$

The quantity \mathcal{T}_{2k+1} is independent of the spin structure because of the fact that an odd-dimensional irreducible complex representation of $\text{SL}(2, \mathbb{C})$ factors through $\text{PSL}(2, \mathbb{C})$. Since $S^3 - K$ has only one cusp, then all spin structures on M_K are acyclic (Corollary 3.4 in [5]). This means that \mathcal{T}_{2k} is also independent of the spin structure (Theorem 7.1 [5]). Thus it is not necessary to consider a spin structure on M_K in our setting. Hence, the above definition may be simplified to the following form deleting η .

Definition 4.2.

$$\begin{aligned}\mathcal{T}_{2k+1}(M_K) &:= \frac{\text{tor}(M_K, \rho_{2k+1}, \mathbf{h})}{\text{tor}(M_K, \rho_3, \mathbf{h})} \in \mathbb{C}^*/\{\pm 1\} \\ \mathcal{T}_{2k}(M_K) &:= \frac{\text{tor}(M_K, \rho_{2k})}{\text{tor}(M_K, \rho_2)} \in \mathbb{C}^*/\{\pm 1\}\end{aligned}$$

Note that it is proved that the quotient is independent of the choices \mathbf{h} (Proposition 4.2 in [5]). Then, we can reduce Theorem 7.1 in [5] to the following statement:

Theorem 4.3 (Theorem 7.1 in [5]).

$$\lim_{k \rightarrow \infty} \frac{\log |\mathcal{T}_{2k+1}(M_K)|}{(2k+1)^2} = \lim_{k \rightarrow \infty} \frac{\log |\mathcal{T}_{2k}(M_K)|}{(2k)^2} = \frac{\text{Vol}(K)}{4\pi}.$$

As in Remark 2.2, the sign of the right hand side is plus.

5. PROOF OF THEOREM 1.1

Case 1. Even-dimensional representation ρ_{2k} case.

By Theorem 0.2 in [4], $H^*(M_K, \rho_{2k}) = 0$ for $k \geq 1$. Then, by Theorem 2.5, we can prove that $\text{tor}(M_K, \rho_{2k}) = \text{tor}(M_K, \alpha \otimes \rho_{2k})|_{t=1} = \Delta_{K, \rho_{2k}}(1)$ from the map at the chain level $C_*(M_K, \alpha \otimes \rho_{2k}) \rightarrow C_*(M_K, \rho_{2k})$ induced by evaluation $t = 1$. Then, we have:

$$\mathcal{T}_{2k}(M_K) = \frac{\text{tor}(M_K, \rho_{2k})}{\text{tor}(M_K, \rho_2)} = \frac{\Delta_{K, \rho_{2k}}(1)}{\Delta_{K, \rho_2}(1)} = \mathcal{A}_{K, 2k}(1).$$

Hence we have done in the case of ρ_{2k} in Theorem 1.1: $\lim_{k \rightarrow \infty} \frac{\log |\mathcal{A}_{K, 2k}(1)|}{(2k)^2} = \frac{\text{Vol}(K)}{4\pi}$ by Theorem 4.3.

Case 2. Odd-dimensional representation ρ_{2k+1} case.

Although the idea of the proof is the same as Yamaguchi's one in [10, 11], I think it is worth outlining it here for the convenience of readers. He investigated the case of the adjoint representation of $\text{SL}(2, \mathbb{C})$, which is essentially equivalent to ρ_3 in our setting.

The homology group $H_*(M_K; \mathbb{Z}) = H_0(M_K; \mathbb{Z}) \oplus H_1(M_K; \mathbb{Z})$ has the basis $\{[p], [\mu]\}$, where $[p]$ is the homology class of a point and $[\mu]$ is that of the meridian of K . Further,

$H_1(\partial M_K; \mathbb{Z})$ has the basis $\{[\mu], [\lambda]\}$, where $[\lambda]$ is the homology class of a longitude of K . By Proposition 4.1, we may define $h^1 = i_*([v \otimes \widetilde{\lambda}])$, $h^2 = i_*([v \otimes \widetilde{T}])$ and $\mathbf{h} = \{h^1, h^2\}$.

It is known that M_K collapses to a 2-dimensional CW-complex W with only one vertex. We call φ this deformation. Thus M_K is simple homotopy equivalent to W . It is enough to prove the theorem for W since a Reidemeister torsion is a simple homotopy invariant.

By Proposition 3.5 in [1], we have $H_0(W, \alpha \otimes \rho_{2k+1}) = 0$. Further, we have the next lemma by the same argument as Proposition 7 in [10] or Proposition 3.1.1 in [11].

Lemma 5.1. *For $*$ = 1, 2, we have: $H_*(M_K, \alpha \otimes \rho_{2k+1}) = 0$.*

Proposition 5.2. *$\text{tor}(M_K, \alpha \otimes \rho_{2k+1})$ has a simple zero at $t = 1$. Moreover the following holds:*

$$\text{tor}(M_K, \rho_{2k+1}, \mathbf{h}) = \lim_{t \rightarrow 1} \frac{\text{tor}(M_K, \alpha \otimes \rho_{2k+1})}{t - 1}.$$

Proof. We define the subchain complex $C'_*(W, \rho_{2k+1})$ of the chain complex $C_*(W, \rho_{2k+1})$ by

$$C'_2(W, \rho_{2k+1}) = \text{span}_{\mathbb{C}} \langle v \otimes \widetilde{\varphi(T)} \rangle, \quad C'_1(W, \rho_{2k+1}) = \text{span}_{\mathbb{C}} \langle v \otimes \widetilde{\varphi(\lambda)} \rangle$$

and $C'_i(W, \rho_{2k+1}) = 0$ ($i \neq 1, 2$). Note that v is fixed by $\rho_{2k+1}(G)$, and the boundary operators of $C'_*(W, \rho_{2k+1})$ are zero by the definition. The modules of this subchain complex are lifts of homology groups $H_*(W, \rho_{2k+1})$. Similarly, we define the subcomplex $C'_*(W, \alpha \otimes \rho_{2k+1})$ of $C_*(W, \alpha \otimes \rho_{2k+1})$ by

$$C'_2(W, \alpha \otimes \rho_{2k+1}) = \text{span}_{\mathbb{C}(t)} \langle 1 \otimes v \otimes \widetilde{\varphi(T)} \rangle, \quad C'_1(W, \alpha \otimes \rho_{2k+1}) = \text{span}_{\mathbb{C}(t)} \langle 1 \otimes v \otimes \widetilde{\varphi(\lambda)} \rangle$$

and $C'_i(W, \alpha \otimes \rho_{2k+1}) = 0$ for $i \neq 1, 2$. Since v is an invariant vector of $\rho_{2k+1}(G)$, we have:

$$\begin{aligned} \partial(1 \otimes v \otimes \widetilde{\varphi(T)}) &= 1 \otimes v \otimes \partial(\widetilde{\varphi(T)}) \\ &= 1 \otimes v \otimes (\mu \cdot \widetilde{\varphi(\lambda)}) - 1 \otimes v \otimes \widetilde{\varphi(\lambda)} \\ &= t \otimes \rho_{2k+1}^{-1}(\mu) v \otimes \widetilde{\varphi(\lambda)} - 1 \otimes v \otimes \widetilde{\varphi(\lambda)} \\ &= t \otimes v \otimes \widetilde{\varphi(\lambda)} - 1 \otimes v \otimes \widetilde{\varphi(\lambda)} \\ &= (t - 1)(1 \otimes v \otimes \widetilde{\varphi(\lambda)}) \end{aligned}$$

Thus the boundary operators of $C'_*(W, \alpha \otimes \rho_{2k+1})$ is given by

$$0 \rightarrow C'_2(W, \alpha \otimes \rho_{2k+1}) \xrightarrow{t-1} C'_1(W, \alpha \otimes \rho_{2k+1}) \rightarrow 0.$$

This means that the homology of $C'_*(W, \alpha \otimes \rho_{2k+1})$ is zero.

By the definition, the chain complex $C'_*(W, \rho_{2k+1})$ has the natural basis:

$$\mathbf{c}' = \{v \otimes \widetilde{\varphi(T)}, v \otimes \widetilde{\varphi(\lambda)}\}.$$

Let $C''_*(W, \rho_{2k+1})$ be the quotient of $C_*(W, \rho_{2k+1})$ by $C'_*(W, \rho_{2k+1})$, \mathbf{c}'' a basis of $C''_*(W, \rho_{2k+1})$, and $\bar{\mathbf{c}}''$ a lift of \mathbf{c}'' to $C_*(W, \rho_{2k+1})$. By Lemma 5.1, we can apply Proposition 3.3.1 in [11] to this setting, then we have:

$$\lim_{t \rightarrow 1} \frac{\text{tor}(C_*(W, \alpha \otimes \rho_{2k+1}), \mathbf{1} \otimes \mathbf{c}' \sqcup \mathbf{1} \otimes \bar{\mathbf{c}}'')}{\text{tor}(C'_*(W, \alpha \otimes \rho_{2k+1}), \mathbf{1} \otimes \mathbf{c}')} = \text{tor}(C_*(W, \rho_{2k+1}), \mathbf{c}' \sqcup \bar{\mathbf{c}}'', \mathbf{h}).$$

By the calculation above, we have $\text{tor}(C'_*(W, \alpha \otimes \rho_{2k+1}), \mathbf{1} \otimes \mathbf{c}') = t - 1$, thus we have this proposition. \square

Proof of Theorem 1.1.

By Theorem 2.5 and Lemma 5.1, we have $\text{tor}(M_K, \alpha \otimes \rho_{2k+1}) = \Delta_{K, \rho_{2k+1}}(t)$. We also have $\Delta_{K, \rho_{2k+1}}(t) = (t-1)\tilde{\Delta}_{K, \rho_{2k+1}}(t)$ and $\text{tor}(M_K, \rho_{2k+1}, \mathbf{h}) = \tilde{\Delta}_{K, \rho_{2k+1}}(1)$ by Proposition 5.2, where $\tilde{\Delta}_{K, \rho_{2k+1}}(t)$ is a rational function. Then,

$$\mathcal{A}_{K, 2k+1}(1) = \frac{\tilde{\Delta}_{K, \rho_{2k+1}}(1)}{\tilde{\Delta}_{K, \rho_3}(1)} = \frac{\text{tor}(M_K, \rho_{2k+1}, \mathbf{h})}{\text{tor}(M_K, \rho_3, \mathbf{h})} = \mathcal{T}_{2k+1}(M_K).$$

Thus we have Theorem 1.1 by Theorem 4.3. □

6. SOME CALCULATIONS ON THE FIGURE EIGHT KNOT COMPLEMENT

Let K be the figure eight knot 4_1 . Note that it is known that the volume of K is $2.02988 \dots$. The knot group $G(K)$ has the following presentation:

$$G(K) = \langle a, b \mid ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b \rangle,$$

where a and b correspond to the meridians of K . Consider the representation of this fundamental group:

$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix},$$

where u is a complex value satisfying $u^2 + u + 1 = 0$. This representation is the holonomy representation of $G(K)$. By the definition, we have $p(\rho(a)^{-1} \begin{pmatrix} x \\ y \end{pmatrix}) = p \begin{pmatrix} x-y \\ y \end{pmatrix}$, and $(x-y)^2 = x^2 - 2xy + y^2$, $(x-y)y = xy - y^2$. Hence, we have:

$$\rho_3(a) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

By the same calculations, we have :

$$\rho_3(b) = \begin{pmatrix} 1 & u & u^2 \\ 0 & 1 & 2u \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_4(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \quad \rho_4(b) = \begin{pmatrix} 1 & u & u^2 & u^3 \\ 0 & 1 & 2u & 3u^2 \\ 0 & 0 & 1 & 3u \\ 0 & 0 & 0 & 1 \end{pmatrix}, \dots$$

Set $A = \rho_2(a) = {}^t\rho(a)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $B = \rho_2(b) = {}^t\rho(b)^{-1} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Via Fox's calculus for $G(K)$, we obtain the denominator of $\Delta_{K, \rho_2}(t) = \det(tB - I) = (t-1)^2$. On the other hand, the numerator of $\Delta_{K, \rho_2}(t) = \det(I - t^{-1}AB^{-1}A^{-1} + AB^{-1}A^{-1}B - tB + BAB^{-1}A^{-1}) = \frac{1}{t^2}(t-1)^2(t^2 - 4t + 1)$. Here we use the value $u = \frac{-1+\sqrt{-3}}{2}$. Continuing in this way, we have obtained the following data.

$$\Delta_{K, \rho_2}(t) = \frac{1}{t^2}(t^2 - 4t + 1), \quad \Delta_{K, \rho_3}(t) = -\frac{1}{t^3}(t-1)(t^2 - 5t + 1)$$

$$\Delta_{K, \rho_4}(t) = \frac{1}{t^4}(t^2 - 4t + 1)^2, \quad \Delta_{K, \rho_5}(t) = -\frac{1}{t^5}(t-1)(t^4 - 9t^3 + 44t^2 - 9t + 1),$$

$$\frac{4\pi \log |\mathcal{A}_{K,4}(t)|}{4^2} = \frac{\pi \log |t^2 - 4t + 1|}{4} \xrightarrow{t=1} \frac{\pi \log 2}{4} \approx 0.544397 \dots$$

$$\frac{4\pi \log |\mathcal{A}_{K,5}(t)|}{5^2} = \frac{4\pi \log \left| \frac{t^4 - 9t^3 + 44t^2 - 9t + 1}{t^2 - 5t + 1} \right|}{5^2} \xrightarrow{t=1} \frac{4\pi \log \frac{28}{3}}{5^2} \approx 1.12273 \dots$$

n	$\frac{4\pi \log \mathcal{A}_{K,n}(1) }{n^2}$	n	$\frac{4\pi \log \mathcal{A}_{K,n}(1) }{n^2}$
6	1.35850...	7	1.58331...
8	1.66441...	9	1.76436...
10	1.79618...	11	1.85105...
12	1.86678...	13	1.90158...
14	1.91009...	15	1.93361...

These calculations were done by using Wolfram Mathematica.

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